Towards the control of transitional flows: a machine-learning perspective

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Joint work with Cedric Beaume, Kuan Li, Steven Tobias

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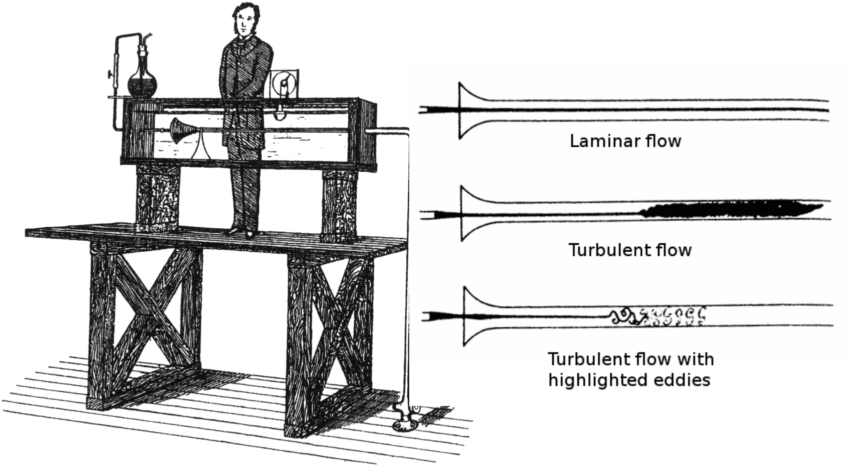
Reynolds experiment



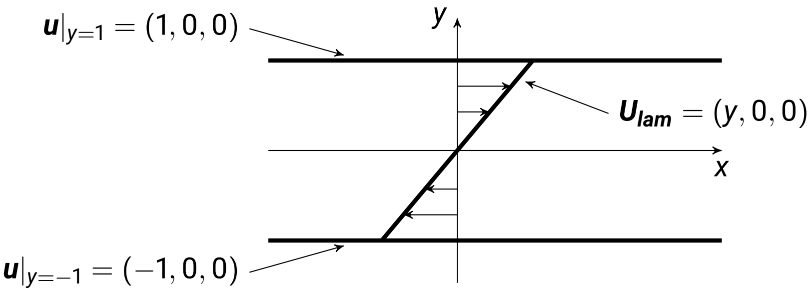
Reynolds, Phil. Trans. R. Soc. London, 174 (1884)

Plane Couette flow

Navier–Stokes equation:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u}$$

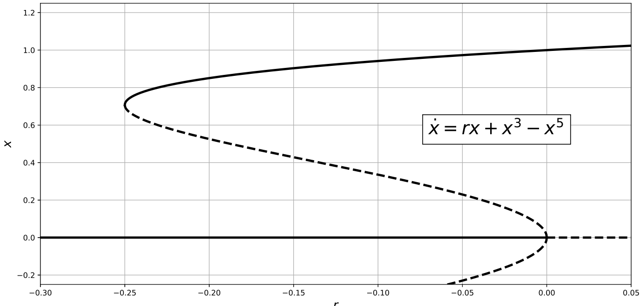
Incompressibility condition: $\nabla \cdot \boldsymbol{u} = 0$
Streamwise and spanwise directions: periodic BCs
Wall-normal direction: no-slip BCs



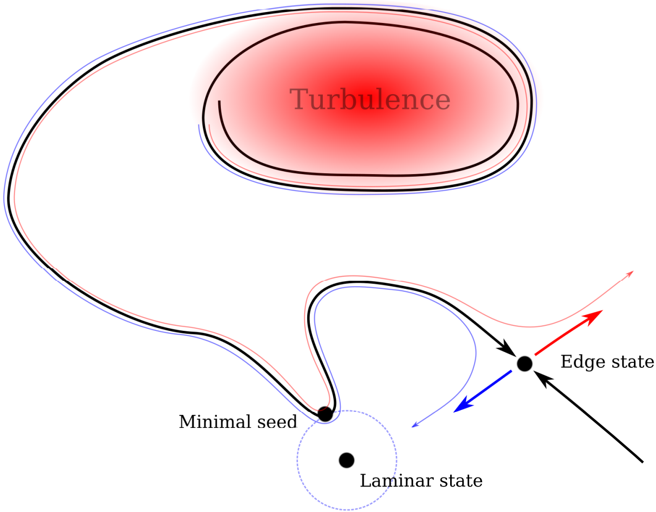
Subcritical transitional flows

	Linearly stable laminar state	Sustained turbulence
Plane Couette flow	all Re	$Re \gtrsim 325$
Pipe flow	all Re	$Re \gtrsim 2040$
Plane Poiseuille flow	$Re \lesssim 5772$	$Re \gtrsim 840$

Transition is complicated by the coexistence of two attractive states:



Edge of chaos is wrapped up around the turbulent saddle[1]

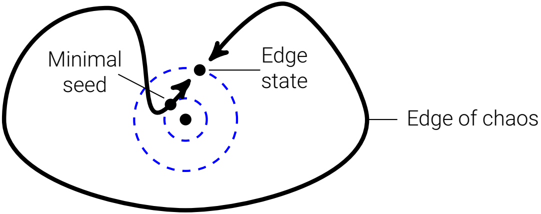


[1]Chantry *et al.*, J. Fluid Mech. **747** (2014)

How robust is the laminar state to perturbations?

Indicators of stability:

- Infinitesimal perturbations $\Longrightarrow$ linear growth rate
- Finite-amplitude perturbations $\Longrightarrow$ the size of the basin of attraction



Laminarisation probability $P_{lam}(E)$ is the probability that a random finite perturbation of energy E laminarises

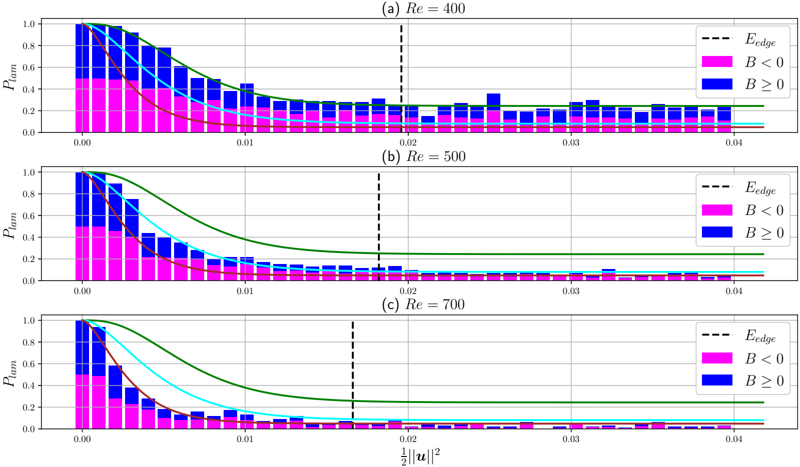
Random perturbation:

$$\boldsymbol{u} = A\boldsymbol{u}_{\perp} + B\boldsymbol{U}_{lam},$$

where $A, B, \boldsymbol{u}_{\perp}$ are generated randomly and $\langle \boldsymbol{u}_{\perp}, \boldsymbol{U}_{lam} \rangle = 0$

Laminarisation probability

- $P_{lam}(E)$ approximates the size of the basin of attraction
- Laminarisation probability fitting: $p(E) = 1 - (1-a)\gamma(\alpha, \beta E)$
- Control strategies can be assessed by comparing $P_{lam}(E)$

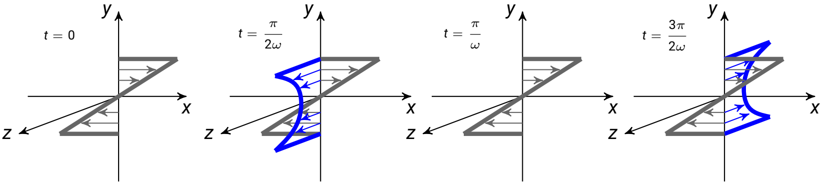


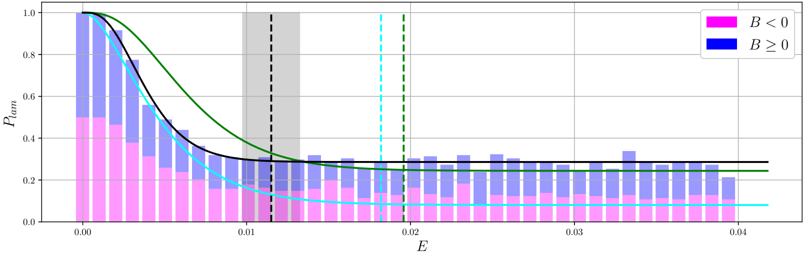
Control strategy: wall oscillations

We impose in-phase oscillations on the walls[2]:

$$\boldsymbol{u}(x, \pm 1, z, t) = \pm \boldsymbol{e_x} + W_{osc} sin(\omega t) \boldsymbol{e_z}$$

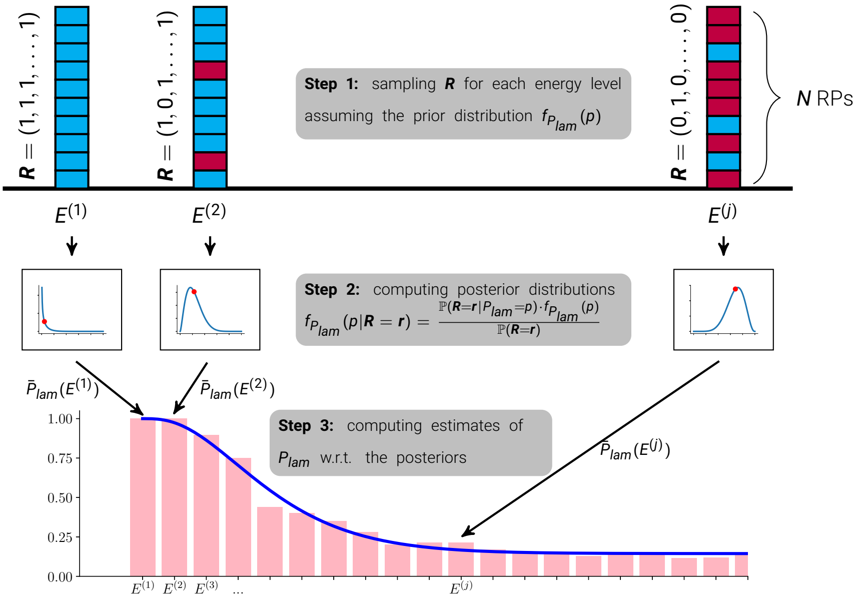
$$\Longrightarrow \quad \boldsymbol{U_{lam}} = y\boldsymbol{e_x} + W(y,t)\boldsymbol{e_z}.$$





[2] Motivated by Rabin *et al.*, J. Fluid Mech. **738** (2014)

Bayesian inference of laminarization probability

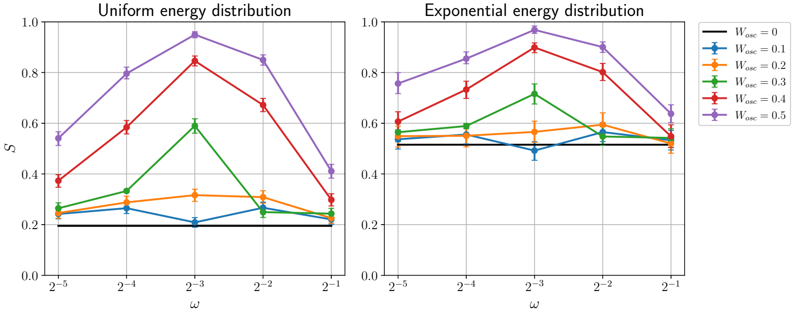


Laminarization score

- Now we can estimate the laminarization score S:

$$S = \int_0^{E_{max}} p(E) f_E(E) dE,$$

- It is assumed that the perturbation energy is distributed as $f_E(E)$
- This is an efficient method for the assessment of laminar flow robustness for a wide range of control parameter values[3]



[3] Pershin, Beaume and Tobias, *submitted, arXiv:2108.07629* (2021)

Learning transition to turbulence via reservoir computing

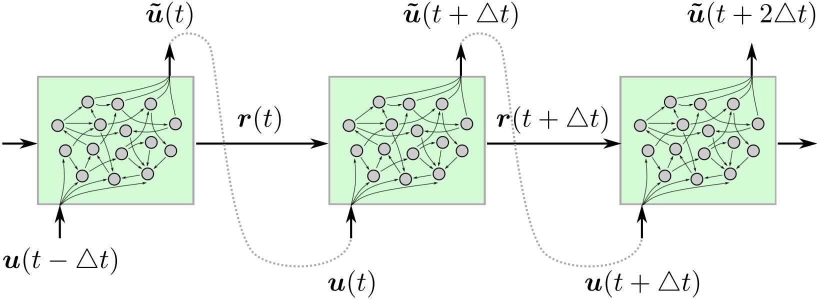
Echo State Network (ESN)

Echo State Network is a reservoir-computing architecture:

$$\boldsymbol{r}(t+\Delta t) = \tanh(\boldsymbol{b} + \boldsymbol{W}_{in}\boldsymbol{u}(t) + \boldsymbol{W}\boldsymbol{r}(t)) + \xi Z,$$
$$\tilde{\boldsymbol{u}}(t+\triangle t) = \boldsymbol{W}_{out}\boldsymbol{r}(t+\Delta t).$$

where

- $\boldsymbol{W}_{in}$ and $\boldsymbol{W}$ are random sparse matrices
- $\boldsymbol{b}$ is a random bias
- Z is a random variable and ξ is the noise strength

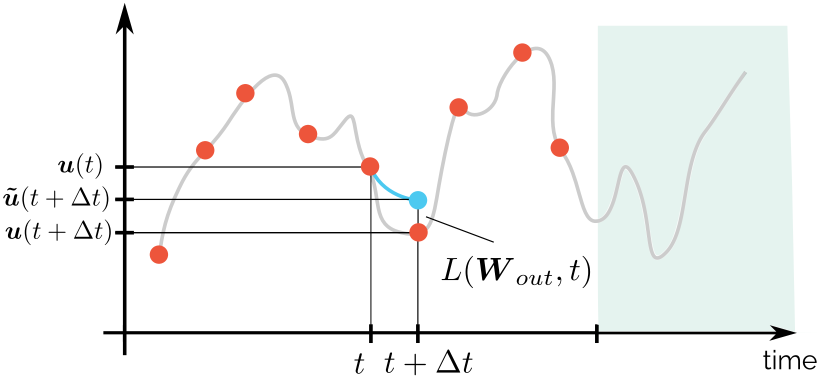


Training

Minimization of the residual sum of squares (RSS):

$$\min_{\boldsymbol{W}_{out}} \sum_{k=1}^{N_t} ||\boldsymbol{W}_{out}\boldsymbol{r}(k\Delta t) - \boldsymbol{u}(k\Delta t)||_2^2.$$

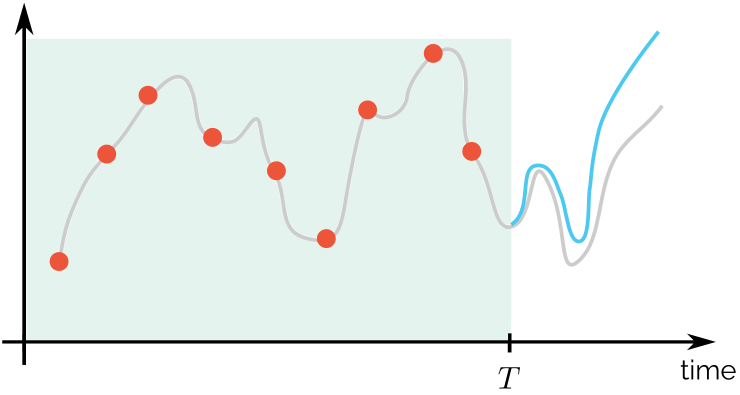
Solution for $\boldsymbol{W}_{out}$ is found via the normal equation.



Prediction mode:

$$\boldsymbol{r}(t+\Delta t) = \tanh(\boldsymbol{b} + \boldsymbol{W}_{in}\tilde{\boldsymbol{u}}(t) + \boldsymbol{W}\boldsymbol{r}(t)) + \xi Z,$$
$$\tilde{\boldsymbol{u}}(t+\triangle t) = \boldsymbol{W}_{out}\boldsymbol{r}(t+\Delta t).$$

We still need to specify the initial condition $\boldsymbol{r}(T)$.



Prediction

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$$\boldsymbol{r}(t+\Delta t) = \tanh(\boldsymbol{b} + \boldsymbol{W}_{in}\boldsymbol{u}(t) + \boldsymbol{W}\boldsymbol{r}(t)) + \xi Z,$$
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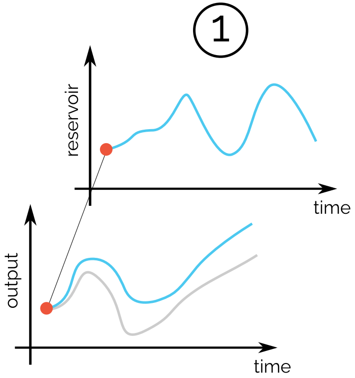
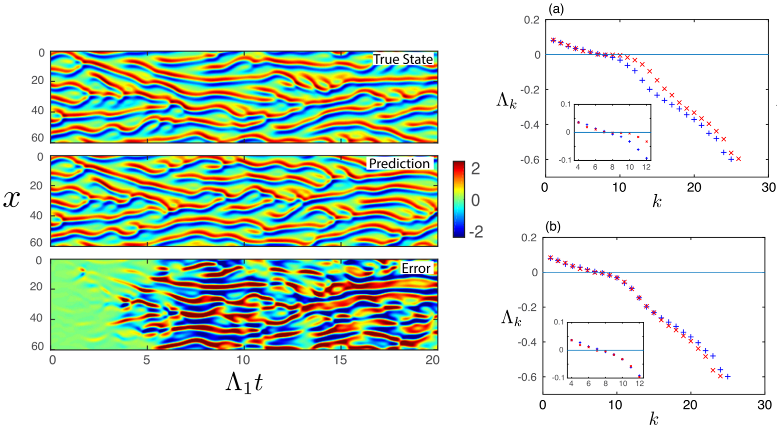


Successful applications

- Low-order dynamical models, Lorenz 63, Lorenz 96
- Kuramoto–Sivashinsky equation
- 2D turbulent Rayleigh–Bénard convection



Pathak *et al.*, Chaos **27**, 121102 (2017)

Moehlis–Faisst–Eckhardt model

The model is obtained by Galerkin projection[4]:

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{j=1}^{9} a_j(t)\boldsymbol{u}_j(\boldsymbol{x}).$$

9-dimensional system of ODEs:

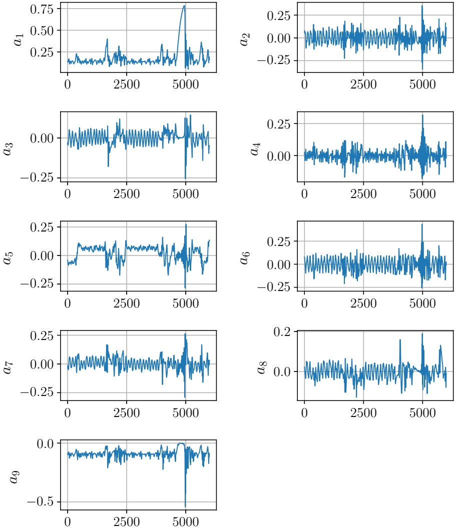
$$\frac{d}{dt}\boldsymbol{a} = \boldsymbol{f}(\boldsymbol{a}; Re, \Gamma_x, \Gamma_z),$$

where $\boldsymbol{a}(t) = [a_1(t), \ldots, a_9(t)]^T$.

Parameters:

- Domain wavelengths: $\Gamma_x = 1.75\pi, \Gamma_z = 1.2\pi$
- Reynolds number: $Re \in [200; 500]$

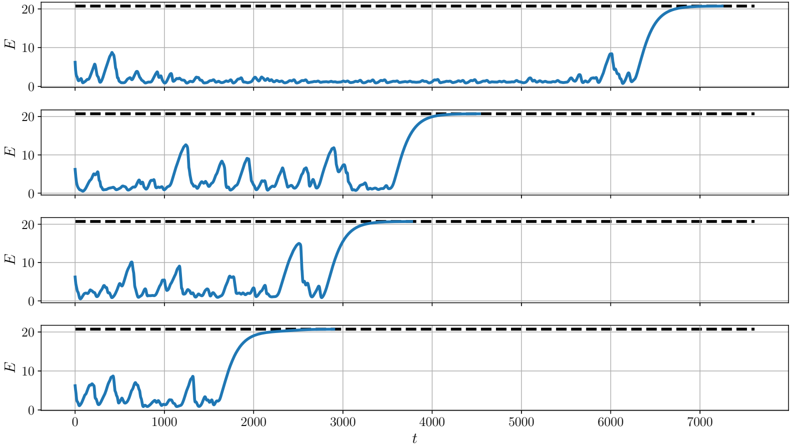
Sustained turbulence: $Re \gtrsim 320$



[4] Moehlis *et al.*, New J. Phys., **6** 56 (2004)

Laminarization ($Re = 300$)

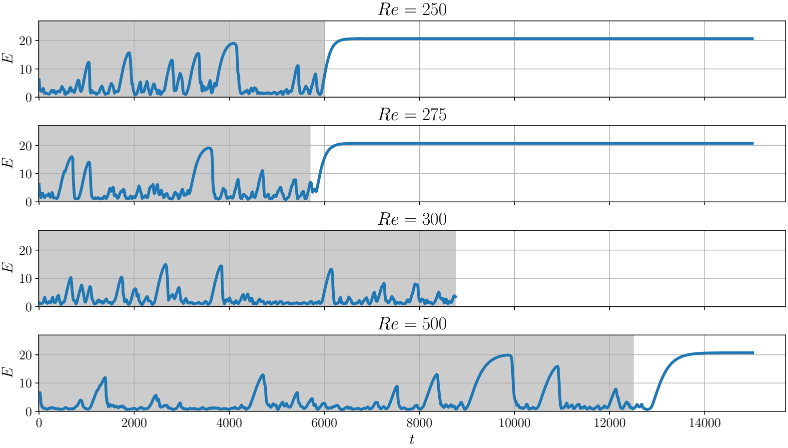
- Turbulence in shear flows is a “leaky” attractor[5]
- As a result, all trajectories eventually end up with laminarization



[5] Avila *et al.*, Science **333**, 6039 (2011)

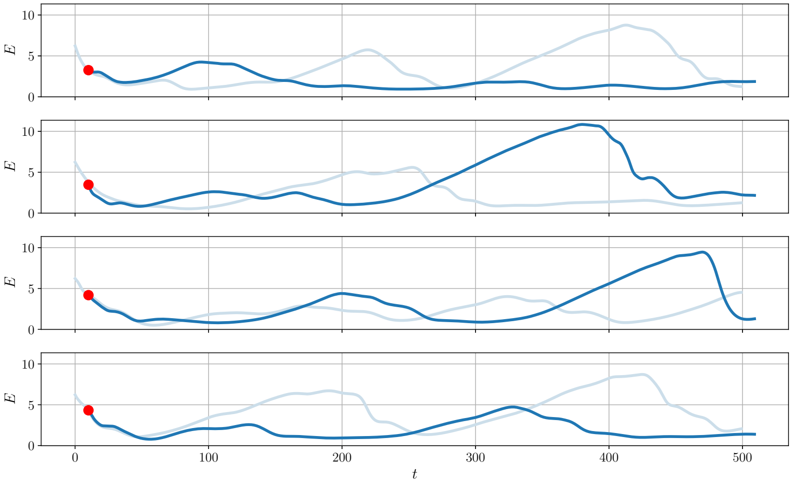
Trajectories used for training

For training, we consider only turbulent trajectories without laminarization events



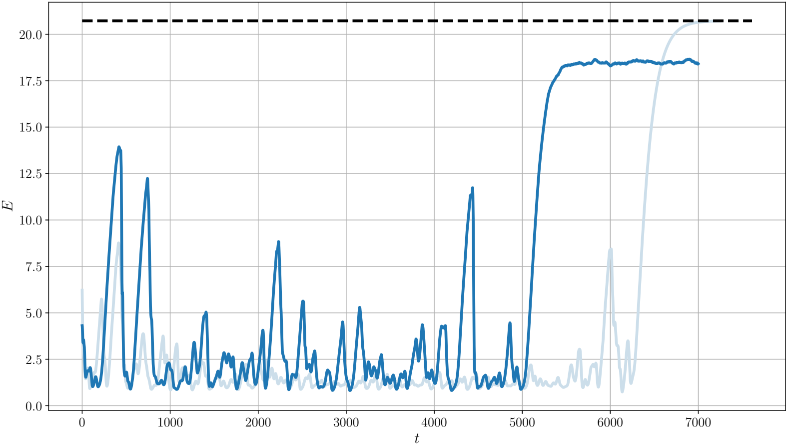
Short-term prediction ($Re = 300$)

Due to the chaotic nature of the original model, the ESN skill for short-term prediction is limited



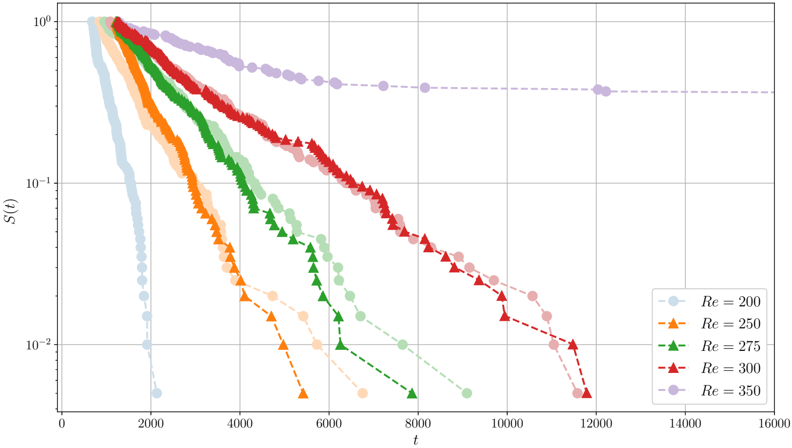
Long-term prediction ($Re = 300$)

- ESN is able to “learn” the laminarization dynamics without experiencing laminarization during the training
- Moreover, ESN is able to replicate the laminar solution



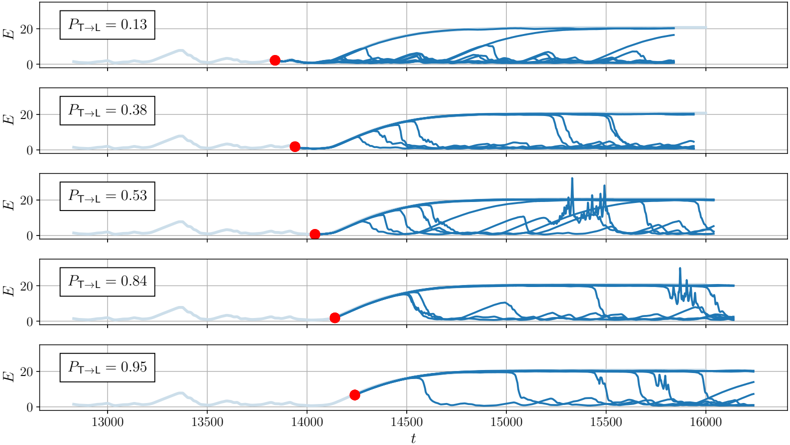
Lifetime distribution

- ESN can successfully replicate the lifetime statistics
- Its skill may degrade depending on the time series used for training



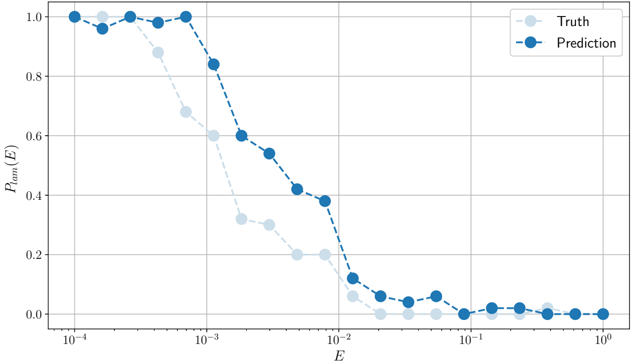
Turbulent-to-laminar transition ($Re = 500$)

- Ensemble approach can be used to estimate the probability of turbulent-to-laminar transition
- The probability grows as the initial condition gets closer to the laminarization event



Laminar-to-turbulent transition ($Re = 500$)

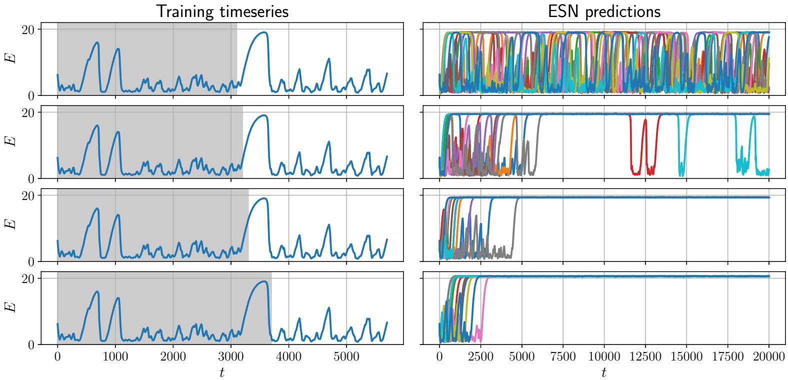
- Robustness of the laminar state to finite-amplitude perturbations is important for assessing laminar-to-turbulent transition
- **Laminarization probability** is the probability that a random perturbation decays as a function of its kinetic energy[6]
- ESN can successfully replicate the laminarization probability



[6] Pershin, Beaume, Tobias, J. Fluid Mech. **895**, A16 (2020)

How can ESN learn laminar dynamics?

- ESN can be expected to embed attractors of the true system by Echo State Network Approximation Theorem [7]
- In practice, it is important to guarantee that the training timeseries includes excursion close to the laminar state



[7] Hart *et al.*, Neural Networks **128**, 234–247 (2020)

Conclusion

Laminarization probability:

- Helps analyse finite-amplitude instabilities
- Approximates the size of the basin of attraction
- Allows to quantify and compare the efficiency of control strategies
- Bayesian inference provides an efficient framework for the laminarization probability estimation
- Minimal seeds and edge states may be misleading for control design

Pershin, Beaume and Tobias, J. Fluid Mech. **895**, A16 (2020)
Pershin, Beaume and Tobias, *submitted, arXiv:2108.07629* (2021)

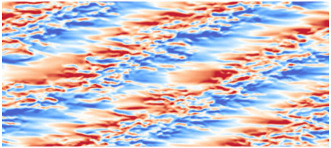
Echo State Networks for transition to turbulence:

- Build a surrogate model of a shear flow
- Able to learn laminar dynamics without seeing it during the training
- Able to approximate key statistics of a transitional flow based only on a single turbulent trajectory
- Perspective: they can be used for designing optimal control

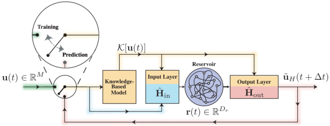
Pershin, Beaume, Li and Tobias, *in preparation* (2021)

Future work

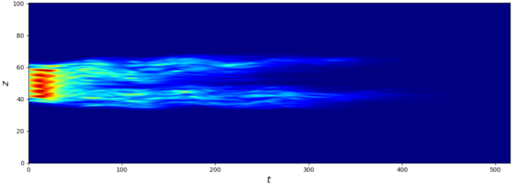
(a) **Spatially extended flows?**[8]



(b) **Physics-informed ESN?**[9]



(c) **Optimal control using reservoir computing?**



[8] Chantry *et al.*, J. Fluid Mech. **791**, R8 (2016)
[9] Pathak *et al.*, Chaos **28**, 041101 (2018)